

FRACTIONAL BOXICITY OF GRAPHS

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Abstract. In this note, we define the *fractional boxicity* of a graph that is a lower bound for the boxicity of the graph.

A *box* in Euclidean k -space is the Cartesian product $I_1 \times I_2 \times \cdots \times I_k$, where I_j is a closed interval on the real line. The *boxicity* of a graph G , denoted by $\text{box}(G)$, is the minimum nonnegative integer k such that G can be isomorphic to the intersection graph of a family of boxes in Euclidean k -space. The concept of boxicity of graphs was introduced by Roberts [3]. It has applications in some research fields, for example, a problem of niche overlap in ecology (see [4] in detail). So far many researchers have attempted to calculate or bound boxicity of graphs with specific structure. Recently, Adiga et al. [1] presented a lower bound for the boxicity of a graph as in **Lemma 1** below. The purpose of this note is to review the lower bound for boxicity in the context of fractional graph theory.

In this note, all graphs are finite, simple and undirected. We use $V(G)$ for the vertex set of a graph G and $E(G)$ for the edge set of the graph G . These notations are also used for hypergraphs. The cardinality of a set X is denoted by $|X|$. A graph is said to be *interval* if its boxicity is at most 1. The symbol \overline{G} denotes the complement of a graph G .

Lemma 1 ([1], Lemma 3.1). The inequality $\text{box}(G) \geq \frac{|E(\overline{G})|}{|E(I_{\min})|}$ holds for a graph G , where I_{\min} is an interval supergraph of G with $V(I_{\min}) = V(G)$ and with the minimum number of edges among all such interval supergraphs of G .

We present a fractional analogue of boxicity that is a lower bound for the boxicity of a graph. A few concepts and results about (hyper)graphs are needed. A graph is said to be *cointerval* if its complement is an interval graph. A *cointerval edge covering* of a graph G is a family \mathcal{C} of cointerval subgraphs of G such that each edge of G is in some graph of \mathcal{C} . The following is a basic result on boxicity.

Theorem 1 ([2], Theorem 3). Let G be a graph. Then, $\text{box}(G) \leq k$ if and only if there exists a cointerval edge covering \mathcal{C} of \overline{G} with $|\mathcal{C}| = k$. Hence $\text{box}(G) = \min\{|\mathcal{C}| \mid \mathcal{C} \text{ is a cointerval edge covering of } \overline{G}\}$.

Our key idea for the definition of a fractional analogue of boxicity is in the way to define a hypergraph associated with a graph. For a graph G , we

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define the hypergraph \mathcal{H}_G as follows:

$$V(\mathcal{H}_G) = E(\overline{G}) \text{ and}$$

$$E(\mathcal{H}_G) = \{E \subset E(\overline{G}) \mid E \text{ induces a cointerval subgraph of } \overline{G}\}.$$

We always assume that hypergraphs associated with graphs are the previous ones. Let \mathcal{C} be a family of hyperedges of a hypergraph \mathcal{H} and write $\mathcal{C} = \{X_1, X_2, \dots, X_k\}$. The family \mathcal{C} is a covering of \mathcal{H} if $V(\mathcal{H}) \subseteq X_1 \cup X_2 \cup \dots \cup X_k$ holds. Note that a covering of \mathcal{H}_G corresponds to a cointerval edge covering of \overline{G} . Hence the covering number of the hypergraph \mathcal{H}_G , the minimum cardinality of a covering of \mathcal{H}_G , is equal to the boxicity of the graph G .

For a graph G , let e_i be an edge of \overline{G} and E_j a hyperedge of \mathcal{H}_G . Moreover let $M = (m_{ij})$ be the incidence matrix of \mathcal{H}_G whose rows are indexed by all edges of \overline{G} and whose columns are indexed by all cointerval subgraphs of \overline{G} , that is, m_{ij} is equal to 1 if $e_i \in E_j$, otherwise 0. Write $E(\mathcal{H}_G) = \{E_1, E_2, \dots, E_n\}$. Let \mathcal{C} be a family of hyperedges in $E(\mathcal{H}_G)$ and $\mathbf{x} = {}^t(x_1, x_2, \dots, x_n) \in \{0, 1\}^n$ the indicator vector of hyperedges in $E(\mathcal{H}_G)$ that corresponds to the family \mathcal{C} , that is, x_i is equal to 1 if $E_i \in \mathcal{C}$, otherwise 0. We see that \mathcal{C} is a cointerval edge covering of \overline{G} if and only if $M\mathbf{x} \geq \mathbf{1}$ (each coordinate of $M\mathbf{x}$ is at least 1) holds. Hence the boxicity of a graph G can be defined as the (optimum) value of the integer program

$$(IP) \text{ “minimize } {}^t\mathbf{1}\mathbf{x} \text{ subject to } M\mathbf{x} \geq \mathbf{1} \text{ and } \mathbf{x} \in \{0, 1\}^n\text{”},$$

that is,

$$\text{box}(G) = \min\{{}^t\mathbf{1}\mathbf{x} \mid M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \in \{0, 1\}^n\},$$

where $\mathbf{1}$ is a vector of all ones. We relax the condition of the integer program (IP) and consider the linear program

$$(LP) \text{ “minimize } {}^t\mathbf{1}\mathbf{x} \text{ subject to } M\mathbf{x} \geq \mathbf{1} \text{ and } \mathbf{x} \geq 0\text{”}.$$

We define the *fractional boxicity* of a graph G , denoted by $\text{box}_f(G)$, to be the value of (LP), that is,

$$\text{box}_f(G) = \min\{{}^t\mathbf{1}\mathbf{x} \mid M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq 0\}.$$

Hence, $\text{box}_f(G) \leq \text{box}(G)$ holds for a graph G . By the way, in the theory of linear programming, one usually considers the dual program of (LP):

$$(D) \text{ “maximize } {}^t\mathbf{1}\mathbf{y} \text{ subject to } {}^tM\mathbf{y} \leq \mathbf{1} \text{ and } \mathbf{y} \geq 0\text{”}.$$

It is well-known in the theory of linear programming that a linear program and its dual have the same value. So we may consider the value of (D) instead of $\text{box}_f(G)$. We notice that the vector \mathbf{y}_* of all $\frac{1}{e}$'s is a feasible solution of (D), where $e = \max_{E_i \in E(\mathcal{H}_G)} |E_i|$. Hence, $\text{box}_f(G) \geq {}^t\mathbf{1}\mathbf{y}_* = \frac{|E(\overline{G})|}{e}$. We note that this lower bound for fractional boxicity of graphs is identical to the one for boxicity of graphs appeared in **Lemma 1**.

An automorphism of a hypergraph \mathcal{H} is a bijection π on $V(\mathcal{H})$ such that $X \in E(\mathcal{H})$ if and only if $\pi(X) \in E(\mathcal{H})$. A hypergraph \mathcal{H} is *vertex-transitive* (*edge-transitive*) if for every pair (w_1, w_2) of vertices (edges) there exists an automorphism π of \mathcal{H} such that $\pi(w_1) = w_2$ holds. The following theorem is derived from Proposition 1.3.4 in [5].

Theorem 2. For a graph G , the inequalities

$$\text{box}(G) \geq \text{box}_f(G) \geq \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}$$

hold. Especially if \overline{G} is edge-transitive, we have the equality

$$\text{box}_f(G) = \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}.$$

Proof. Note that the fractional boxicity of a graph G is the same concept with the fractional covering of the hypergraph \mathcal{H}_G . It is possible to show that the hypergraph \mathcal{H}_G is vertex-transitive by the edge-transitivity of \overline{G} , so the above equality holds by Proposition 1.3.4 in [5]. The following **Lemma 2** completes the proof of this theorem. \square

Lemma 2. If \overline{G} is edge-transitive for a graph G , the hypergraph \mathcal{H}_G is vertex-transitive.

Proof. For every pair of vertices $e_1, e_2 \in V(\mathcal{H}_G) = E(\overline{G})$, there exists an automorphism $\pi : V(\overline{G}) \rightarrow V(\overline{G})$ such that $\pi(e_1) = e_2$ holds by our assumption. We can check that π induces a bijection $\overline{\pi}$ on $E(\overline{G})$ in a natural way: $\overline{\pi}(uv) = \pi(u)\pi(v)$. Moreover E is in $E(\mathcal{H}_G)$ if and only if $\overline{\pi}(E)$ is in $E(\mathcal{H}_G)$ since π and its inverse π^{-1} map a subgraph H of \overline{G} to the subgraph isomorphic to H . Hence the map $\overline{\pi}$ is the desired one. \square

We take another way to reach the fractional boxicity of graphs. Let s be a positive integer. The s -fold boxicity of a graph G , denoted by $\text{box}_s(G)$, is the minimum cardinality of a multiset $\{E_1, E_2, \dots, E_k\}$ of cointerval subgraphs of \overline{G} such that each edge of \overline{G} is in at least s cointerval subgraphs in the multiset. Note that $\text{box}_1(G) = \text{box}(G)$. Since the subadditivity $\text{box}_{s+t}(G) \leq \text{box}_s(G) + \text{box}_t(G)$ holds for a graph G and $s, t \geq 1$, the following limit exists and we have the following equality by **Lemma 3**:

$$\lim_{s \rightarrow \infty} \frac{\text{box}_s(G)}{s} = \inf \left\{ \frac{\text{box}_s(G)}{s} \mid s \geq 1 \right\}.$$

Lemma 3. Let \mathbb{Z}^+ and \mathbb{R}^+ be the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. If $g : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is subadditive, that is, $g(m+n) \leq g(m) + g(n)$ holds for any $m, n \in \mathbb{Z}^+$, the limit $\lim_{m \rightarrow \infty} \frac{g(m)}{m}$ exists and is equal to $\inf \frac{g(m)}{m}$. \square

A basic result on the fractional covering numbers of hypergraphs guarantees $\text{box}_f(G) = \lim_{s \rightarrow \infty} \frac{\text{box}_s(G)}{s}$ (see Theorem 1.2.1 in [5]).

Example 1. We present an example of a graph G that satisfies $\text{box}(G) = \text{box}_f(G) > \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}$. Let K_{n_1, n_2, \dots, n_k} be a complete k -partite graph with $n_1 > n_2 \geq \dots \geq n_k$ and write $G = K_{n_1, n_2, \dots, n_k}$. It is known that the boxicity of K_{n_1, n_2, \dots, n_k} is equal to the number of classes each of which has at least two elements. For simplicity, we assume that $\text{box}(K_{n_1, n_2, \dots, n_k}) = k$, that is, $n_k \geq 2$. Then it is easy to see that $\text{box}_s(K_{n_1, n_2, \dots, n_k}) = sk$. Note

that $\max_{E_i \in E(\mathcal{H}_G)} |E_i|$ is equal to $\binom{n_1}{2}$. Hence we have

$$\lim_{s \rightarrow \infty} \frac{\text{box}_s(G)}{s} = k > \frac{\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2}}{\binom{n_1}{2}} = \frac{|E(\overline{G})|}{\max_{E_i \in E(\mathcal{H}_G)} |E_i|}.$$

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REFERENCES

- [1] A. Adiga, L. S. Chandran and N. Sivadasan, Lower bounds for boxicity, *Combinatorica* 34 (2014) 631-655.
- [2] M. B. Cozzens and F. S. Roberts, Computing the boxicity of a graph by covering its complement by cointerval graphs, *Discrete Appl. Math.* 6 (1983) 217-228.
- [3] F. S. Roberts, On the boxicity and cubicity of a graph, in: *Recent Progress in Combinatorics*, Academic Press, New York (1969) 301-310.
- [4] F. S. Roberts, *Discrete mathematical models, with applications to social, biological, and environmental problems*, Prentice-Hall, New Jersey, 1976.
- [5] E. R. Scheinerman and D. H. Ullman, *Fractional Graph Theory, A rational approach to the theory of graphs*, Dover Publications, Inc. Mineola, New York, 2011.

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